Fermat's Principle & Snell's law 18.9.21

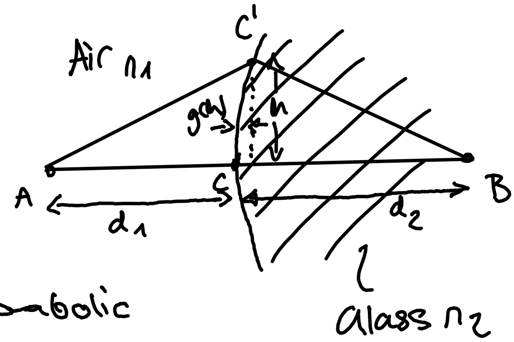
Speed of light is lower in materials compared to vacuum. $v \leq c$
The Index of refraction is thereby given by $n = c/v \geq 1$ and depending on the material.
Optical path lenth is $D = nd$
Fermat's principle tells us, that light follows the path with the lowest optical path lenth. $\frac{dD(\vec{r})}{dp} \stackrel{!}{=} 0$
In homogenous materials light follows straight lines.
Snell's law tells us, that $n_2 \sin\theta_2 = n_1 \sin\theta_1$, where θ_i is the angle between the Surface-Normal and the light Ray.

Lenses

22.9.25

Lenses are optical devices used to focus as much light as possible to one point using Fermats principle.

Single surface lenses



$$D_{AC'B} \overset{!}{=} D_{ACB}$$

using $(1+\varepsilon)^\alpha \approx 1+\alpha\varepsilon$, $|\varepsilon| \ll 1$

we find, that

$$g(h) = \frac{h^2}{2(n_2-n_1)}\left(\frac{n_1}{d_1}+\frac{n_2}{d_2}\right) \sim \text{parabolic}$$

in reality we make another approximation and use not parabolic lenses, but spherical, which are much easier to manufacture.

so $g(h) = R - \sqrt{R^2-h^2} \approx \frac{h^2}{2R}$ for $\frac{h^2}{R^2} \ll 1$

where R is the radius of curvature.

Often we look at far away objects $d_1 \to \infty$, that we want to focus onto a focal point $d_2 = f$.

In that case

$$R = \frac{f(n_2-n_1)}{n_2} \qquad f = \frac{R n_2}{n_2-n_1} \qquad \frac{n_2}{f} = \left(\frac{n_1}{d_1}+\frac{n_2}{d_2}\right)$$

if $f<0$ we call that a virtual image.

Note, that because of the approximations we will get aberrations.

Two surface lenses

Lensmakers formula $\frac{1}{f} = (n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = \frac{1}{d_1}+\frac{1}{d_2}$, for very thin lenses

$f>0$ converging, $f<0$ diverging, $d_1>0$ real object, $d_1<0$ virtual

$d_2>0$ real image, $d_2<0$ virtual image

Ray-tracing

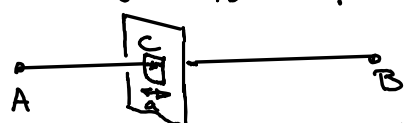
1. Incoming Rays go through center
2. Incoming Parallel will go in a line that goes through the focus
3. Rays passing through focus, will be parallel

Magnification factor $M = \frac{h'}{h} = -\frac{d_2}{d_1}$

25.9.23

Wave Optics, Huygens vs. Fermat, Fraunhofer diffraction

- Using Maxwell's equations and rotrot = grad div − Δ we get $\vec{\nabla}^2\vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$, $\vec{\nabla}^2\vec{B} = \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}$
  ↳ Superpositions hold
  ↳ monochromatic Waves : $E(\vec{r},t) = U(\vec{r})\, e^{-i\omega t}$
  ↳ Plane Wave : $E(\vec{r},t) = E_0 e^{i(\vec{k}\vec{r}-\omega t)}$
  ↳ spherical Wave : $E(\vec{r},t) = \frac{A}{r} e^{i(kr-\omega t)}$
- Huygens's principle : every point of wavefront ~~sets~~ interpreted as source of new hemispherical wave that propagates in forward direction (normal).
- We're applying Huygens principle to the following setup



we see that $U(C) \propto \frac{1}{|\vec{r}_A - \vec{r}_C|} e^{ik|\vec{r}_A - \vec{r}_C|}$

and that $U(B) \propto U(C) \frac{1}{|\vec{r}_B - \vec{r}_C|} e^{ik|\vec{r}_B - \vec{r}_C|}$

using superposition we get $U(B) \propto \iint \frac{1}{|r_C - r_A||r_C - r_B|} e^{ik(|r_A - r_C| + |r_C - r_B|)}$

Then we make some approximations

(1) $a \ll r_A, r_B$, (2) Taylor approx, (3) Fraunhofer (flat Wavefronts)
(4) drop linear terms

We get $U(B) \propto \frac{a^2 e^{ik(r_A + r_B)}}{r_A r_B} \operatorname{sinc}\left(\frac{ka}{2}\left(\frac{x_A}{r_A} + \frac{x_B}{r_B}\right)\right) \operatorname{sinc}\left(\frac{ka}{2}\left(\frac{y_A}{r_A} + \frac{y_B}{r_B}\right)\right)$

, where $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$

Note Intensity is $I \propto |U(r)|^2 \propto a^4$

↳ $\lambda \ll a \Rightarrow$ Ray optics i.e. same as Fermat
↳ Fraunhofer diffraction $(s \gg \frac{a^2}{\lambda})$ — 2D-Fourier-Transform

$U(x,y) \propto \iint t(x,y)\, e^{-i(k_x x - k_y y)}\, dx\, dy$, where $t(x,y)$ is transmission depending on x,y

Rayleigh-criterion, Abbés limit, Birefringece 29.9.25

- Rayleigh-criterion tells us if two points can be resolved. We need the max of the diffraction pattern from first object to be at least at the first minimum of the second one.

$$\boxed{\sin\varphi = \frac{\lambda}{a}}$$, where φ is the incoming angle

- Abbés limit tells us, that to see a pattern, we need at least first diffraction mins. For that to be the case

$$\boxed{d > \frac{\lambda}{2NA}}$$, where $NA = n\sin\alpha$, where α is max angle the lense can accept.

- Polarization

(1) Linear: $\vec{E}(\vec{r},t) = (E_x\hat{x} + E_y\hat{y})e^{i(kz-\omega t)}$

(2) Elliptical and circular: $\vec{E}(\vec{r},t) = (E_x\hat{x} + E_y\hat{y}e^{i\phi})e^{i(kz-\omega t)}$

$\phi = 0 \rightsquigarrow$ linear

$\phi = \frac{\pi}{2} \rightsquigarrow$ Right-hand-circular-polarization σ^+

$\phi = -\frac{\pi}{2} \rightsquigarrow$ Left-hand-circular-polarization σ^-

- Birefringece: Effect in almost all real material that is $n_x \neq n_y$, where n_i is the index of refraction in i-th comp.

$n_x > n_y \Rightarrow n_x$ is slow axis, n_y fast axis

$\vec{E}(\vec{r},t) = (E_x\hat{x}e^{ik_xz} + E_y\hat{y}e^{i(k_yz+\phi)})e^{-i\omega t}$

if we want to change ϕ we use $(n_x - n_y)L \cdot k_0 = \phi$, i.e. Half-Wave-Plates

p/s-polarization, Maxwell in Mat., IOR, Fresnel eq.
Brewsters Angle
3.10.25

- p-polarization is when $E \parallel$ Plane of wave direction and Normal to Surface
s-polarization is when $E \perp$ "
- Maxwell in Material (no free charge / current)
(1) $\vec{\nabla} \cdot \vec{D} = 0$ (3) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
(2) $\vec{\nabla} \cdot \vec{B} = 0$ (4) $\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$
in s.c. linear materials $\vec{B} = \mu_0 \mu_r \vec{H}$, $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$
where non-magnetic materials fullfill: $\mu_r = 1$
- From $\Delta \vec{E} = \varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2}{\partial t^2} \vec{E}$ we get $v = \frac{1}{\sqrt{\varepsilon_r \varepsilon_0 \mu_r \mu_0}} = \frac{c}{n}$
where the IOR is $n = \sqrt{\varepsilon_r \mu_r}$
- We can use Gausses Law (Zylinder with top half in Vacum and bottom in Mat) to see $H_{\parallel 1} = H_{\parallel 2}$ and $E_{\parallel 1} = E_{\parallel 2}$
using this we get Fresnels equations
  - p-polarized light
$$t_p = \frac{E_t}{E_i} = \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$
$$r_p = \frac{E_r}{E_i} = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t}$$
  - s-polarized light
$$t_s = \frac{E_t}{E_i} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$
$$r_s = \frac{E_r}{E_i} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$
- We know frequencies are material invariant.
- Brewster angle (no reflection) $\theta_B = \arctan\left(\frac{n_2}{n_1}\right)$

Dispersion, Gratings, spectrally resolvable, Michelson, Spectral density, 6.10.25

- a Material has different IOR depending on the frequencies of the waves. ⇒ Prism.
- Grating Spectrometer: Periodic setup of absorbing/transmitting and reflective material.
  ↳ we have to consider interference pattern which is why incoming angle ≠ outgoing angle

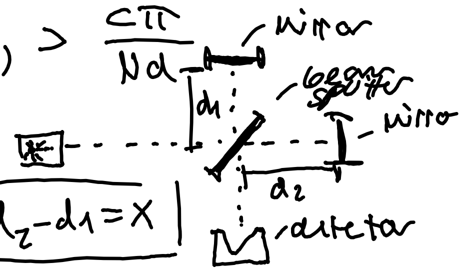
We find $I \propto |U|^2 \propto \dfrac{\sin^2(N\phi/2)}{\sin^2(\phi/2)} \Rightarrow$ max at $\phi/2 = m\pi$

- smallest freq. diff we can resolve is

$$\Delta\omega = (m_2 - m_1)c = \frac{2\pi c}{Nd(\sin\theta_i - \sin\theta_j)} > \frac{c\pi}{Nd}$$

- Interferometers: Michelson

$$I = |U|^2 = \frac{I_0}{2}\left(1+\cos(2\omega x/c)\right) \qquad \boxed{d_2 - d_1 = x}$$



↳ function of x and λ

Intensity between ω and $\omega + d\omega$ is given by $I_0 S(\omega) d\omega$ then $I(x) = \frac{I_0}{2}\int_0^\infty S(\omega)\left(1+\cos(2\omega x/c)\right) d\omega$

In reality we cannot have ∞ but x_{max}, so

$$S(\omega) \propto \int_0^{x_{max}} (2I(x) - I_0)\cos(2\omega x/c)\, dx$$

$$\propto \operatorname{sinc}\left(\frac{(\omega - \omega_0)x_{max}}{c}\right) + \operatorname{sinc}\left(\frac{(\omega+\omega_0)x_{max}}{c}\right)$$

↳ Spectral resolution with rayleigh: $\Delta\omega = \dfrac{\lambda\omega}{2x_{max}}$

Balmer Series, Rydberg, Photoelectric effect, Planck const.

9.10.25

- We can determine what types of elements we're looking at by shining light on them and analyzing the reflected frequencies.
- Balmer series / Rydberg try to describe the pattern for Hydrogen. They found

$$f = \frac{c}{\lambda} = c \cdot R_\infty \left(\frac{1}{n'^2} - \frac{1}{n^2}\right), \text{where } n' \leq n \in \mathbb{N}$$

and the Rydberg constant $R_\infty = 1.097 \cdot 10^7 \, m^{-1}$

- Photoelectric effect
  - ↳ Light causes <u>negative</u> charges to be ejected
  - ↳ Light does <u>not</u> cause positive charges to be ejected
  - ↳ Electrons aren't ejected if frequencie is too low.

We found

$$E_{max} = hf - \Phi, \text{ where } h = 0.004136 \, eV/THz$$

ist Planks constant and Φ the work-function

Intensity only changes how many electrons get ejected. Not the Energy they have.

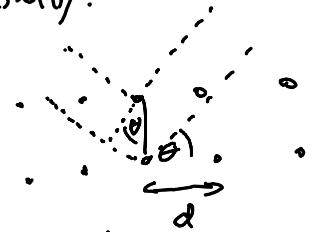
- Einstein: Packets of Energy $hf \rightsquigarrow$ quanta aka. Photons
- Photoelectric effect does <u>not</u> prove existence of Photons, since we'd get the same result for quantum-mechanical description of Atoms.
- Photoelectric Effect works in reverse. This is how we get X-Ray

$$hf = eV_0 + \Phi \Rightarrow \text{vary } V_0 \text{ to get desired } f.$$

Rayleigh Scattering, Bragg-scattering, De Broglie, 19.10.25
Quantum Wave, k/real-space, Heisenberg uncertainty

∘) Assuming elastic reflection of EM-Waves at surfaces, we drive charges on surface material in oscillation, and since accelerating charges emit radiation we get Rayleigh Scattering $E_{emit} \propto \omega^2 p \propto \frac{\omega^2 e E}{m(\omega^2-\omega_0^2)} \cos\omega t$, where p is dipol-moment, ω_0 is freq of Wave and ω freq of charge oscillation.
For $\omega \ll \omega_0$ we see $I \propto \omega^4$, so higher freqs are scattered more efficiently.

∘) Scattering at Crystall (see right)
we get phase difference $\delta\varphi = \frac{2\pi}{\lambda} 2d\sin\theta$
hence constructive interference at $\lambda = 2d\sin\theta$
which is the Bragg condition



∘) Particles have Wave-like Nature and De-Broglie Wavelength $\lambda = \frac{h}{p}$, where p is momentum of particle and h is plancks constant.

∘) Wavepacket is superposition of Waves with different Wave-Vectors. We have $\omega(k) = \frac{\hbar k^2}{2m}$. For a small Δk we get

$$\psi(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} A e^{i(kx-\omega(k)t)}\, dk \simeq 2A\Delta k\, e^{i(k_0 x-\omega_0 t)} \cdot \mathrm{sinc}\left(\Delta k\left[x-\frac{\hbar k_0}{m}t\right]\right)$$

This is Wave-Equation in real-space, for k-space we take the FT. inv FT to reverse.
↳ Peak of sinc-Envelope is at $x_{peak} = \frac{\hbar k_0}{m} t$
and $\frac{dx_{peak}}{dt} = \frac{\hbar k_0}{m} = v_0$ is Group-Velocity
Probability of Particle beeing in $[x, x+dx]$ is $P(x,t)dx = |\psi(x,t)|^2 dx$
↳ ψ^2 behaves like Prob-dist.

∘) Heisenberg uncertainty: $\Delta p \Delta x \geq \frac{\hbar}{2}$, where Δ are std.
$|\psi(x,t)|^2$ sharp $\Rightarrow$ $|\psi(k,t)|^2$ wide and vice versa

QM, Wave functions, Operators, Change of Bases 23.10.25

- States are in Vectorspace Hilbertspace and denoted by $|S\rangle$
  ↳ ex: Numbers of Apples $|S\rangle = \sum_N a_N |N\rangle \Rightarrow a_N = \langle N|S\rangle$
  $|N\rangle = \langle N|^*$ ~ complex transpose, $P_S(N) = |a_N|^2$
- A Basis $\{|N\rangle\}$ is orthonormal iff $\langle M|N\rangle = \delta_{MN}$
- for Position-Representation with Basis $\{|\phi_n\rangle\}$ where $|\phi_n\rangle$ has Wavefunction $\phi_n(x)$. Orthonormal iff $\int_D \phi_n^* \phi_m dx = \delta_{nm}$
- We call Basis complete, iff for all $\psi(x)$ wavefunctions we can write it as linear combination.

Dirac		Position
$|\psi\rangle = \sum_n \psi_n |\phi_n\rangle$	$\longleftrightarrow$	$\psi(x) = \sum_n \psi_n \phi_n(x)$
$\psi_n = \langle \phi_n|\psi\rangle$	$\longleftrightarrow$	$\psi_n = \int_D \phi_n^*(x)\psi(x)dx$

- For a Basis to be complete we need
  $$\sum_n \phi_n^*(x')\,\phi_n(x) = \delta(x-x')$$
- For continuous Basis $|\phi_\alpha\rangle \longrightarrow \phi(\alpha, x)$
  ↳ Any state $|\psi\rangle = \int \tilde{\psi}(\alpha)\,|\phi_\alpha\rangle\, d\alpha \rightarrow \psi(x) = \int \tilde{\psi}(\alpha)\,\phi(\alpha,x)\,d\alpha$
  Orthonormality $\langle \phi_\alpha|\phi_{\alpha'}\rangle = \delta(\alpha-\alpha') \rightarrow \int \phi^*(\alpha,x)\,\phi(\alpha',x)\,dx = \delta(\alpha-\alpha')$
- Observables are Quantities we can observe. It can be expressed as a Operator. (Hermitian)
  ↝ ex $\hat{N}|N\rangle = N|N\rangle$
- expectation Value $\langle \hat{N}\rangle_S = \langle S|\hat{N}|S\rangle = \sum_N N P_N$
- momentum operator $\hat{p} \rightsquigarrow -i\hbar \frac{\partial}{\partial x}$, $\hat{P}|\psi\rangle = p|\psi\rangle$
  Energy operator $\hat{E} \rightsquigarrow i\hbar \frac{\partial}{\partial t}$, $\hat{E}|\psi\rangle = E|\psi\rangle$
- $\psi(x,t) = \langle x|\psi(t)\rangle$, $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

not really clear yet...

Unitary-Operator, Measurements, Commutator,
Schrödinger-Equation, Hamiltonian, time independent S.E.
inf. Square-Well Eigenenergys, Parity-Operator 2.11.25

- For any orthonormal Basis $\{|\phi_n\rangle\}$ $\hat{I} = \mathbb{1} = \sum_n |\phi_n\rangle\langle\phi_n|$
  or for continous $\{|\phi(x)\rangle\}$ $\mathbb{1} = \int |\phi(x)\rangle\langle\phi(x)| dx$
- We can use the above to see $\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{ipx/\hbar} \tilde{\Psi}(p,t) dp$
  and $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ is the position rep of $|p\rangle$
- Eigenstates of observable $\hat{O}$ form a complete Basis
  $\{|O_n\rangle\}$ or $\{|O(\alpha)\rangle\}$, where $\hat{O}|O_n\rangle = O_n|O_n\rangle$, $\hat{O}|O(\alpha)\rangle = O(\alpha)|O(\alpha)\rangle$
  Any State can be written as $|\Psi\rangle = \sum_n c_n(t)|O_n\rangle$ or
  $|\Psi\rangle = \int \Psi(\alpha,t)|O(\alpha)\rangle d\alpha$
- Making a measurement will get you a Eigenvalue
- Prob. of getting O_n is $P = |\langle O_n|\Psi\rangle|^2 = |c_n|^2$
  or between $O(\alpha)$ and $O(\alpha+d\alpha)$ is $P = |\langle O(\alpha)|\Psi\rangle|^2 d\alpha = |\Psi(\alpha,t)|^2 d\alpha$
  ↳ After measurement system/Wavefunction collapses to $|O_n\rangle$
  or to $|\Psi'\rangle$, when $\Psi'(x,t)$ is sharply peaked at $O(\alpha)$
- We write $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = 0$ if $\hat{A}$ and $\hat{B}$ commute
- Schrödinger-Eq: $i\hbar \frac{\partial \Psi}{\partial t} = \underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}}_{\text{kin. E}} + \underbrace{\hat{U}(x)\Psi}_{\text{pot. E}}$
  with Hamiltonian
  $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \hat{V} = \frac{\hat{p}^2}{2m} + \hat{V}$
  we can write S.E as $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$
- For time independent $|\Psi(x,t)|^2$ we get T.I.S.E:
  $\hat{H}|\Psi_n\rangle = E_n|\Psi_n\rangle$, when E_n is the Eigenenergy
- Eigenenergys for particle in Infinit-Square-Well are
  $E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$
- Parity operator $\hat{\pi}\Psi(y) = \Psi(-y)$, it holds $[\hat{\pi},\hat{H}] = 0$
  so $\hat{H}$ and $\hat{\pi}$ share orthonormal Basis and
  $\lambda_{\hat{\pi}} \in \{\pm 1\}$, when $+1$ is for even and -1 for odd fct.

Finite Square Well, QHO, Creation/annihilation

- Finite Square well is characterized by $V(x) = \begin{cases} V_0, & |x| \geq a \\ 0 & |x| < a \end{cases}$
  ↳ We can use symmetry to split
  even solutions $\frac{\sqrt{k_0^2 - k^2}}{k} = \tan\left(\frac{ka}{2}\right)$
  odd solutions $\frac{\sqrt{k_0^2 - k^2}}{k} = -\cot\left(\frac{ka}{2}\right)$, $\cot = \frac{1}{\tan}$
  where $k = \sqrt{\frac{2mE}{\hbar^2}}$, $k_0 = \sqrt{\frac{2mV_0}{\hbar^2}}$
- for $E < V_0$ we have non zero ψ-values outside of well which is Quantum tunneling
- For $E < V_0$ we have $\sin(kx)/\cos(kx)$ standing waves with exponential decays outside the well.
- $E > V_0$ we have ref-coef: $|r|^2 = \left|\frac{k-q}{k+q}\right|^2$
  $q = \sqrt{k^2 - k_0^2}$ trans-coef: $|t|^2 = \left|\frac{2k}{k+q}\right|^2$
- QHO with $V = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$ and res. freq $\omega = \sqrt{\frac{k}{m}}$
  ↳ using first order taylor-Approximation
  $\hat{X} = \sqrt{\frac{m\omega}{\hbar}}\hat{x}$, $\hat{P} = \sqrt{\frac{1}{\hbar m\omega}}\hat{p}$, when $\hat{H} = \frac{\hbar\omega}{2}(\hat{X}^2 + \hat{P}^2)$
- Annihilation $\hat{a} = \frac{1}{\sqrt{2}}(\hat{X} + i\hat{P})$, Creation $\hat{a}^+ = \frac{1}{\sqrt{2}}(\hat{X} - i\hat{P})$
  ↳ $[\hat{a}, \hat{a}^+] = 1$ and $\hat{H} = \hbar\omega(\hat{a}^+\hat{a} + \frac{1}{2})$
- Suppose we have Eigenstate $|n\rangle$ with $\hat{a}^+\hat{a}|n\rangle = n|n\rangle$
  ↳ Eigenenergies of QHO are $E_n = \hbar\omega(n + \frac{1}{2})$
- if we have $|n\rangle$ we can get $|n+1\rangle$ and $|n-1\rangle$
  $|n+1\rangle = \frac{\hat{a}^+|n\rangle}{\sqrt{n+1}}$ $|n-1\rangle = \frac{\hat{a}|n\rangle}{\sqrt{n}}$, $|n\rangle = \frac{(\hat{a}^+)^n|0\rangle}{\sqrt{n!}}$
  with $\Delta x = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{2n+1}$, $\Delta p = \sqrt{\frac{\hbar m\omega}{2}}\sqrt{2n+1}$
  ↳ Ground state $|0\rangle$ is $\hat{a}|0\rangle = 0$
- Phonons are quantized Energy levels of QHO

S.E. in 3D, Legendre-Functions, Spherical Harmonics, Coulomb-Potential, Hydrogen 18.11.25

- S.E. in 3D has $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$, where $\hat{H} = \frac{1}{2m}|\hat{\vec{p}}|^2 + V(\hat{\vec{r}})$ where $\hat{\vec{r}} = (\hat{x}, \hat{y}, \hat{z})$, $\hat{\vec{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$, $\hat{p}_i = -i\hbar \frac{\partial}{\partial_i}$, $\hat{\vec{p}} = -i\hbar \nabla$
  we can write $i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\hat{\vec{r}})\Psi$
- Different components commute: $[\hat{x}, \hat{p}_y] = [\hat{y}, \hat{p}_x] \dots = 0$
  but same don't: $[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = \dots = i\hbar$
- $\int_{\mathbb{R}^3} |\Psi(\vec{r}, t)|^2 dxdydz = 1 \Rightarrow |\Psi(\vec{r}, t)|^2 dxdydz$ is Prob of finding particle in $dxdydz$ at $\vec{r}$
- Using spherical coordinates (r, θ, φ) and ∇^2 in spherical we use Ansatz $\Psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$, $Y(\theta, \varphi) = T(\theta)\Phi(\varphi)$
  we find $\Phi(\varphi) = e^{im\phi}$, $m \in \mathbb{Z}$
  $T(\theta) = A P_\ell^m(\cos\theta)$, where P_ℓ^m are associated Legendre-fct which is rather complicated. $\ell \in \mathbb{N}_0 \Rightarrow Y(\theta, \varphi) = Y_\ell^m(\theta, \varphi)$ which are called spherical harmonics for Quantum Numbers m, ℓ.
- The Radial part is heavily depending on $V(\vec{r})$ so we can only discuss cases individually.
- Coulomb-Potential $V(r) = \frac{-e^2}{4\pi\varepsilon_0} \frac{1}{r}$ (f.e. in Hydrogen)
  $\hookrightarrow$ we only discuss $E<0$ (electrons are Bound!)
- We get Eigenenergies $E_n = -\frac{m_e e^4}{2\hbar^2 (4\pi\varepsilon_0)^2 n^2}$ $E_1 \approx -13.6eV = -R_c$ - Rydberg
  and Bohr-Radius $a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2}$, "most probable Radius"
- $\Psi_{n\ell m}(r, \theta, \varphi) = R_{n\ell}(r) Y_\ell^m(\theta, \varphi)$, where n is principle Q.N. and $\ell = 0, 1 \dots n-1$, $m = -\ell, -\ell+1, \dots \ell-1, \ell$
- Degeneracy is number of multiple states with same Eigenenergy and given by $d(n) = \sum_{\ell=0}^{n-1} (2\ell+1) = n^2$

TO-DO